S-rings on compact transformation groups.

by G.V. Wood

1. Introduction

S-rings in finite groups appeared as early as 1933 due to Schur in [3], but their theory was not developed until 1949 and subsequently by Wielandt and Tamaschke. ([5], [4] et al.) Only recently have they been considered in continuous groups.

For a compact group G, an S-ring is a subalgebra of C(G) with respect to both pointwise and convolution multiplications, which is self-adjoint with respect to both involutions, and which contains the constant functions. In this paper, we consider the compact group G acting on a homogeneous space Ω by right multiplication and show that the centraliser algebra has a strongly dense subalgebra which is the set of convolution operators by functions in a generalised S-ring. This is an S-ring in the case where Ω can be identified with a subgroup H of G - e.g. a quotient group or a direct product component. After preliminary material in §2, we consider in §3 the particular case when G is the direct product of two subgroups H and K, as this gives a clearer motivation. Finally in §4, we deal with the general case which has some interesting variations. This work generalises results of Wielandt in [6] p.80.

2. Definitions and preliminary results.

Let G be a compact group with identity 1, and C(G) the set of continuous complex-valued functions on G. We shall use the following notation: for $f, g \in C(G)$, $x, y \in G$, $f_y(x) = f(xy)$; ${}_yf(x) = f(y^{-1}x)$; $f^*(x) = \overline{f(x^{-1})}$; $f\#(x) = \overline{f(x)}$; $(f * g)(x) = \int_G f(y)g(y^{-1})dm_G(y)$, where $m_G(.)$ denotes Haar measure on G. By the invariance of m_G, we have

PROPOSITION 2.1 For $f, g \in C(G)$ and $x \in G$, ${}_x(f * g) = {}_xf * g$ and $(f * g)_x = f * g_x$.

The pair (G, Ω) will denote a compact group and a homogeneous space upon which G acts. i.e. Ω is a compact Hausdorff space and there exists a continuous map $\Omega \times G \to \Omega$ $\{(\alpha, x) \longmapsto \alpha^x\}$ satisfying:

a) $(\alpha^x)^y = \alpha^{xy}$ all $\alpha \in \Omega$, $x, y \in G$

b) $\alpha^1 = \alpha$ all $\alpha \in \Omega$

c) For each α, the map $G \to \Omega$ $(x \longmapsto \alpha^x)$ is an open map.

Throughout this paper we will assume that the action is transitive. i.e. for each α, $\{\alpha^x : x \in G\} = \Omega$. We can identify Ω with the space of right cosets of a subgroup K where $K = \{x : \alpha_0^x = \alpha_0\}$ some fixed $\alpha_0 \in \Omega$. We shall denote the coset space of K by G/K, and c) ensures that it has the identification topology. If L is a closed subgroup of G, then L also acts on Ω, but not in general transitively. Denote by $C(\Omega)$ the set of continuous complex-valued functions on Ω, and by $C(\Omega, L)$ those functions in $C(\Omega)$ which are constant on the orbits of L. i.e. the

sets $\{\{\alpha^x : x \in L\} : \alpha \in \Omega\}$. Let $\mathfrak{B}$ be the set of bounded linear operators on the Banach space $C(\Omega)$, and clearly we can embed G in $\mathfrak{B}$ as follows: For each $x \in G$, let $X \in \mathfrak{B}$ be defined by $(X\xi)(\alpha) = \xi(\alpha^x)$ where $\xi \in C(\Omega)$ and $\alpha \in \Omega$. Then obviously we have:

PROPOSITION 2.2 $\xi \in C(\Omega, L)$ if and only if $X\xi = \xi$ all $x \in L$.

We now give the two basic definitions.

DEFINITION 2.3 The centraliser algebra of (G, Ω), denoted by $\mathfrak{B}(G, \Omega)$, is the set $\{T \in \mathfrak{B} : TX = XT \text{ all } x \in G\}$.

DEFINITION 2.4 A subset $\mathfrak{F}$ of $C(G)$ is an S-ring of G if it is a pointwise subalgebra and a convolution subalgebra of $C(G)$, * and # - self-adjoint, and contains the constant functions. We denote the constancy sets of $\mathfrak{F}$ by $\{G_\lambda : \lambda \in \Lambda\}$ with G_0 the set containing 1. G_0 is a subgroup and $G_\lambda = G_0 G_\lambda G_0$. (see [7]) If $G_0 = \{1\}$, $\mathfrak{F}$ is called unitary. Any subset of G which is the union of G_λ's is called a Λ-set, and a function constant on the G_λ's is called a Λ-function. Clearly if $\mathfrak{F}$ is uniformly closed, $\mathfrak{F}$ is the set of all Λ-functions. We need the following theorem and corollary from [7].

THEOREM 2.5 Let $\mathfrak{F}$ be a closed (= uniformly closed) S-ring on G. Then for each $f \in \mathfrak{F}$, there is an approximate identity in $\mathfrak{F}$.

COROLLARY 2.6 Let $\mathfrak{F}$ be a closed unitary S-ring on G. Then for each $f \in C(G)$, there is an approximate identity in $\mathfrak{F}$.

We will need the following strengthened form of 2.5.

THEOREM 2.7 Let $\mathfrak{F}$ be a closed S-ring on G with constancy sets $\{G_\lambda : \lambda \in \Lambda\}$. If $f \in C(G)$ and is constant on the right (left, double) cosets of G_0, then there is a left (right, two-sided) approximate identity for f in $\mathfrak{F}$. i.e. given $\varepsilon > 0$, there exists a Λ-neighbourhood V of G_0 and $u \in \mathfrak{F}$ such that $u(x) \geq 0$, $u(x) = 0$ off V, $\int u(x)dm_G(x) = 1$, and for any such $u \in \mathfrak{F}$, $||u * f - f||_\infty < \varepsilon$ ($||f * u - f||_\infty < \varepsilon$, $||u * f - f||_\infty < \varepsilon$ and $||f * u - f||_\infty < \varepsilon$).

Proof* We will prove the theorem for the case when $f \in C(G)$ is constant on the right cosets of G_0. The other cases are proved similarly. For such a function f, ${}_xf = f$ all $x \in G_0$. Since f is uniformly continuous on G, given $\varepsilon > 0$, there exists a neighbourhood N of 1 such that $||{}_xf - {}_yf|| < \varepsilon$ all $xy^{-1} \in N$. The family $\{Nx : x \in G_0\}$ covers G_0 and by compactness, there exists $x_1, \ldots x_n \in G_0$ such that the family $\{Nx_i : i = 1, 2 \ldots n\}$ covers G_0. Let $N_1 = \bigcup_1^n Nx_i$. Then N_1 is an open neighbourhood of G_0. If $V = \cup\{G_\lambda : G_\lambda \subset N_1\}$, then V is an open Λ-neighbourhood of G_0. ([2] p.125). We have $y \in V \Longrightarrow y \in N_1 \Longrightarrow yx_i^{-1} \in N$ some i

$$\Longrightarrow ||{}_yf - {}_{x_i}f||_\infty < \varepsilon$$
$$\Longrightarrow ||{}_yf - f||_\infty < \varepsilon \text{ since } x_i \in G_0.$$

By normality, there is a non-negative function $u \in \mathfrak{F}$ such that $u(x) = 0$ off V, $u(x) \neq 0$ on G_0, and by taking a suitable multiple, we can assume that $\int_G u(x)dm_G(x) = 1$. For any such u,

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* This proof is essentially the same as for Theorem 4.1 in [7], but it is repeated here in its more precise form for completeness.

$$|(u * f)(x) - f(x)| = |\int_G u(y)f(y^{-1}x)dm_G(y) - f(x)|$$
$$= |\int_G u(y)\{f(y^{-1}x) - f(x)\}dm_G(y)|$$
$$\leq \int_G u(y)|f(y^{-1}x) - f(x)|dm_G(y)$$
$$\leq \sup_{y \in V} ||_y f - f||_\infty < \varepsilon.$$

This is true for all x, and so $||u * f - f||_\infty < \varepsilon$ and the theorem is proved.

3. The case where G is a direct product.

Suppose $G = HK = KH$, $H \cap K = \{1\}$ where H, K are closed subgroups. Then we can identify Ω with H. We will show first that C(H, K) is a closed S-ring on H. For $f \in C(H)$, denote by $\tilde{f}$ its extension to G by making it constant on the right cosets of K. i.e. $\tilde{f}(x) = f(1^x)$ all $x \in G$. This gives a 1-1 correspondence between C(H) and those functions in C(G) which are constant on the right cosets of K.

LEMMA 3.1 If $f \in C(H)$ and $x \in G$, Xf is the restriction to H of $\tilde{f}_x$.

Proof For all $y \in H$, $(Xf)(y) = f(y^x) = f(1^{yx}) = \tilde{f}(yx) = \tilde{f}_x(y)$.

LEMMA 3.2 If $f \in C(H, K)$, $g \in C(H)$ then $\tilde{f} * \tilde{g} = \widetilde{f * g}$

(Note: the first convolution is over G, the second over H).

Proof Since both of the functions $\tilde{f} * \tilde{g}$ and $\widetilde{f * g}$ are constant on the right cosets of K, it is sufficient to show that they are

equal on H. For $x \in H$,

$$(\tilde{f} * \tilde{g})(x) = \int_G \tilde{f}(xy)\tilde{g}(y^{-1})dm_G(y)$$
$$= \int_K\int_H \tilde{f}(xzt)\tilde{g}(t^{-1}z^{-1})dm_H(z)dm_K(t) \quad ([1]\ p.\ 132.)$$
$$= \int_K\int_H \tilde{f}(xz)\tilde{g}(z^{-1})dm_H(z)dm_K(t) \quad (\text{since } t \in K)$$
$$= \int_H f(xz)g(z^{-1})dm_H(z)$$
$$= (f * g)(x)$$
$$= (\widetilde{f * g})(x).$$

Hence $\tilde{f} * \tilde{g} = \widetilde{f * g}$.

COROLLARY 3.3 If $f \in C(H, K)$, $g \in C(H)$, $x \in G$, then $X(f * g) = f * Xg$.

Proof Using 3.1, 3.2, 2.1,

$$X(\widetilde{f * g}) = (\widetilde{f * g})_x = (\tilde{f} * \tilde{g})_x = \tilde{f} * \tilde{g}_x = \tilde{f} * \widetilde{Xg} = \widetilde{f * Xg}.$$

Hence $X(f * g) = f * Xg$.

THEOREM 3.4 If L is a closed subgroup of G, then the functions in C(H, K) act by left convolutions on C(H, L). i.e. $f \in C(H, K)$, $g \in C(H, L)$ implies $f * g \in C(H, L)$.

Proof Suppose $f \in C(H, K)$, $g \in C(H, L)$. For $x \in L$ $X(f * g) = f * Xg = f * g$ using 3.3 and 2.2. Hence, by 2.2, $f * g \in C(H, L)$.

COROLLARY 3.5 C(H, K) is a closed unitary S-ring.

Proof C(H, K) is a convolution algebra by putting L = K in 3.4. To show that it is *-self-adjoint, suppose $x, y \in H$. Then x, y are in the same orbit under $K \Longleftrightarrow x \in H \cap KyK$

$\Longleftrightarrow x^{-1} \in H \cap Ky^{-1}K \Longleftrightarrow x^{-1}$, y^{-1} are in the same orbit under K. The other properties of an S-ring are immediate from the definition. That C(H, K) is unitary follows from the fact that $\{1\}$ is an orbit of K.

It is clear that we can embed C(H, K) in $\mathcal{L}$(G, H). For $f \in C(H, K)$ denote by L_f the operator on C(H) mapping g into $f * g$. Then, as a direct corollary to 3.3, we have:

__LEMMA 3.6__ For $f \in C(H, K)$, $L_f \in \mathcal{L}(G, H)$

In [6] p. 80, it was shown that in the finite case, $C(H, K) \cong \mathcal{L}(G, H)$. Clearly this is not true in our case. In fact, the identity operator I is in $\mathcal{L}$(G, H), but not in $\{L_f : f \in C(H, K)\}$. However, on account of 2.5, we know that there is an approximate identity in C(H, K). This suggests that $\{L_f : f \in C(H, K)\}$ is strongly dense in $\mathcal{L}$(G, H). This is what we now prove. We need first a stronger form of 3.4.

__LEMMA 3.7__ If L is a closed subgroup of G, then C(H, L) is an invariant subspace of each $T \in \mathcal{L}(G, H)$.

__Proof__ Let $f \in C(G, L)$, $x \in L$ and $T \in \mathcal{L}(G, H)$. Then $XTf = TXf = Tf$ and so $Tf \in C(G, L)$.

__LEMMA 3.8__ $f, g \in C(H)$, $T \in \mathcal{L}(G, \Omega)$ implies $T(f * g) = Tf * g$.

__Proof__ Suppose $T \in \mathcal{L}(G, H)$. Then, in particular, T commutes with right translation by elements of H. But right convolution by a function in C(H) is the uniform limit of finite sums of right translates. Hence, since T is continuous, $T(f * g) = Tf * g$.

__THEOREM 3.9__ $\{L_f : f \in C(H, K)\}$ is strongly dense in $\mathcal{L}(G, H)$.

Proof Let $T \in \mathcal{L}(G, H)$, and $f_1, f_2 \dots f_n \in C(H)$. Since $C(H, K)$ is a closed unitary S-ring, there exists an approximate identity u_k in $C(H, K)$ for each f_i $(i = 1, \dots, n)$. Thus, for each i, $u_k * f_i \to f_i$ uniformly as $k \to \infty$. Let $g_k = Tu_k$. Then $g_k \in C(H, K)$ by 3.7 and $g_k * f_i = Tu_k * f_i = T(u_k * f_i)$ by 3.8. Therefore, for each $i = 1, \dots, n$,

$$||g_k * f_i - Tf_i||_\infty = ||T(u_k * f_i - f_i)||_\infty \leqslant ||T|| \, ||u_k * f_i - f_i||_\infty$$

$\to 0$ as $k \to \infty$.

i.e. T is in the strong closure of $\{L_f : f \in C(H, K)\}$.

4. The general case.

For an arbitrary homogeneous space Ω, we use essentially the same methods, though there are some interesting differences since Ω has not got a group structure. In particular we show that although $C(\Omega, K)$ is not itself an S-ring, it is the image of an S-ring on G under the canonical map $C(G) \to C(\Omega)$. (Note that in this case also, there is a natural 1-1 correspondence between $C(\Omega)$ and $\{f \in C(G) : f$ constant on the right cosets of $K\}$ given by $\tilde{\xi}(x) = \xi(\alpha_0^x)$ where $\xi \in C(\Omega)$, $\tilde{\xi} \in C(G)$ and α_0 is the element of Ω corresponding to K in G/K). The steps in the proof follow the same pattern as in §3, so we simply quote the corresponding results and give proofs only when there is a significant variation.

LEMMA 4.1 If $\xi \in C(\Omega)$ and $x \in G$, $\widetilde{X\xi} = \tilde{\xi}_x$.

Proof As 3.1.

Define the convolution $\xi * \eta$ of two functions $\xi, \eta \in C(\Omega)$ by

$$\widetilde{\xi * \eta} = \tilde{\xi} * \tilde{\eta}.$$

This is well defined since the convolution in G of two functions constant on the right cosets of K is also constant on the right cosets of K.

REMARK 4.2 This convolution is not equivalent to the usual one in H when $G = HK = KH$, $H \cap K = \{1\}$. Indeed, we will show that with the convolution just defined, $C(\Omega, K)$ is a left ideal of $C(\Omega)$. However, it is true that if $f \in C(H, K)$, then the two definitions of $f * g$ coincide. (Lemma 3.2)

Define an involution in $C(\Omega, K)$ by $\widetilde{\xi^*} = \tilde{\xi}^*$.

REMARK 4.3 This cannot, in general be extended to the whole of $C(\Omega)$ since the set of functions $\{\tilde{\xi} : \xi \in C(\Omega)\}$ is the set of continuous functions on G which are constant on the right cosets of K, and so is not *-self-adjoint (unless K is normal). However it is well-defined on $C(\Omega, K)$ since $\xi \in C(H, K) \Longleftrightarrow \tilde{\xi}$ is constant on the double cosets of K.

LEMMA 4.4 If $\xi, \eta \in C(\Omega)$ and $x \in G$, $X(\xi * \eta) = \xi * X\eta$.

Proof As 3.3, though here we do not need that $\xi \in C(\Omega, K)$.

LEMMA 4.5 If L is a closed subgroup of G, then $C(\Omega, L)$ is a left ideal of $C(\Omega)$.

Proof As 3.4 - again we do not need $\xi \in C(\Omega, K)$.

LEMMA 4.6 $\{\tilde{\xi}:\xi \in C(\Omega, K)\}$ is a closed S-ring on G with constancy set containing 1 equal to K.

Proof We have shown above that $\{\tilde{\xi}:\xi \in C(\Omega, K)\}$ is the double coset S-ring of the subgroup K.

LEMMA 4.7 If $\xi \in C(\Omega)$, $L_\xi \in \mathcal{L}(G, \Omega)$.

Proof As 3.6.

Thus, in this case, we have the whole of $C(\Omega)$ embedded in $\mathcal{L}(G, \Omega)$. However, this is simply because of the different definition of convolution, and it is still true that $\{L_\xi:\xi \in C(\Omega, K)\}$ is strongly dense in $\mathcal{L}(G, \Omega)$, as we shall now prove.

LEMMA 4.8 If L is a closed subgroup of G, then $C(\Omega, L)$ is an invariant subspace of each $T \in \mathcal{L}(G, \Omega)$

Proof As 3.7.

LEMMA 4.9 If $\xi, \eta \in C(\Omega)$, $T \in \mathcal{L}(G, \Omega)$, then $T(\xi * \eta) = T\xi * \eta$.

Proof As 3.8, though here we use $\tilde{\xi}$ and $\tilde{\eta}$ in $C(G)$ and the corresponding operator $\tilde{T}$ which commutes with translation by elements of G.

LEMMA 4.10 For functions in $C(\Omega)$, there is a left approximate identity in $C(\Omega, K)$.

Proof If $\xi_1, \ldots, \xi_n \in C(\Omega)$, then $\tilde{\xi}_1, \ldots, \tilde{\xi}_n \in C(G)$ and are constant on the right cosets of K. By 2.6, there exists a left approximate identity in $\mathfrak{F} = \{\tilde{\xi}:\xi \in C(\Omega, K)\}$. i.e. there exists $u_k \in C(\Omega, K)$ such that for each i, $\tilde{u}_k * \tilde{\xi}_i \to \tilde{\xi}_i$ uniformly as $k \to \infty$. Since $\widetilde{u_k * \xi_i} = \tilde{u}_k * \tilde{\xi}_i$, we have, for each i, $u_k * \xi_i \to \xi_i$ uniformly as $k \to \infty$.

THEOREM 4.11 $\{L_\xi : \xi \in C(\Omega, K)\}$ is strongly dense in $\mathcal{L}(G, \Omega)$.

Proof In the preceding lemmas, we have all the corresponding results to those that were used in 3.9, and so the proof is the same.

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